Solutions to Problem Set 1

1-2 An *edge cover* of a graph $G = (V, E)$ is a subset of R of E such that every vertex of V is incident to at least one edge in R. Let G be a bipartite graph with no isolated vertex. Show that the cardinality of the minimum edge cover R^* of G is equal to $|V|$ minus the cardinality of the maximum matching M^* of G. Give an efficient algorithm for finding the minimum edge cover of G. Is this true also for non-bipartite graphs?

Let $\rho(G)$ be the size of a minimum edge cover and $\nu(G)$ the size of the maximum matching. A maximum matching covers $2\nu(G)$ vertices. Because of the connectedness, the remaining $n - 2\nu(G)$ vertices can be covered by no more than $n - 2\nu(G)$ edges. These edges and the maximum matching thus form an edge cover of size $n - \nu(G)$. On the other hand, a minimum edge cover has to be a forest (an acyclic graph). (Indeed, if it has any cycle then the removal of any edge of the cycle would still give an edge cover, of smaller cardinality.) The number of connected components of this forest is precisely $n - \rho(G)$ because every component of the forest is a tree, and a tree on k vertices has $k - 1$ edges, and one can take one edge per component to get a matching. We therefore have $\nu(G) \geq n - \rho(G)$.

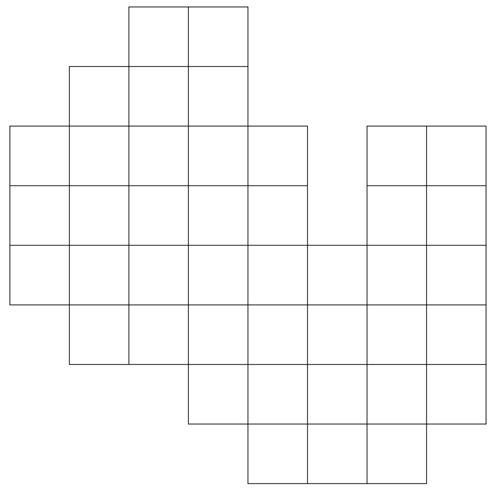
The first part of the proof clearly yields an algorithm for finding a minimum edge cover given an algorithm for finding a maximum cardinality matching.

Yes, the result remains true for non-bipartite graphs. Observe the proof above carries over for non-bipartite graphs.

1-4 Consider the problem of perfectly tiling a subset of a checkerboard (i.e. a collection of unit squares, see example below) with dominoes (a domino being 2 adjacent squares).

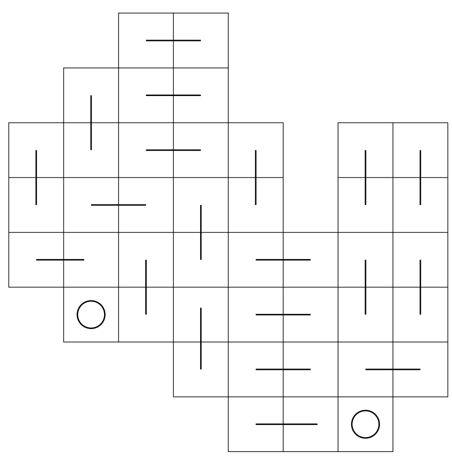
(a) Show that this problem can be formulated as the problem of deciding whether a bipartite graph has a perfect matching.

(b) Can the following figure be tiled by dominoes? Give a tiling or a short proof that no tiling exists.



(a) Consider the bipartite graph G with a vertex for each square and two squares are adjacent if they share an edge. This graph is bipartite since the squares can be colored black and white in a checkerboard pattern.

Any perfect tiling gives a perfect matching by simply selecting the edges corresponding to the dominoes selected. And vice versa.



(b)

Figure 0.1: Maximum configuration of dominoes.

We claim that the configuration shown in Figure 0.1 is a maximum one and so no perfect tiling exists. We will prove that the matching M corresponding to the configuration in Figure 0.1 is maximum by showing that there is no augmenting path as in the lecture. (Alternatively we could use Hall's theorem.)

Let A be the set of black squares and B the set of white squares. Orient the edges of G according to M, i.e. all the edges in M are oriented from B to A, and the edges not in M are oriented from A to B as in Figure 0.3.

Let v be the only exposed vertex of A and w be the only exposed vertex of B, and consider L to be the set of vertices reachable from v (the enclosed area in Figure 0.3). Since w is not in L we obtain that no augmenting path exists.

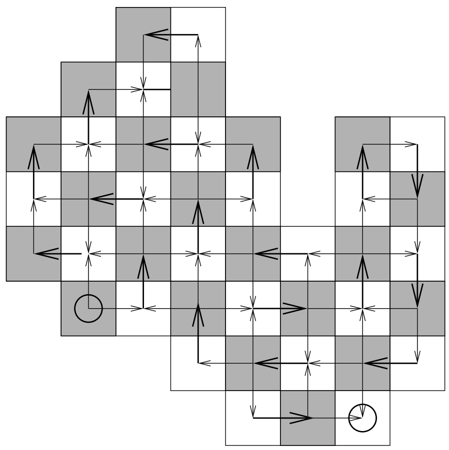


Figure 0.2: Oriented graph.

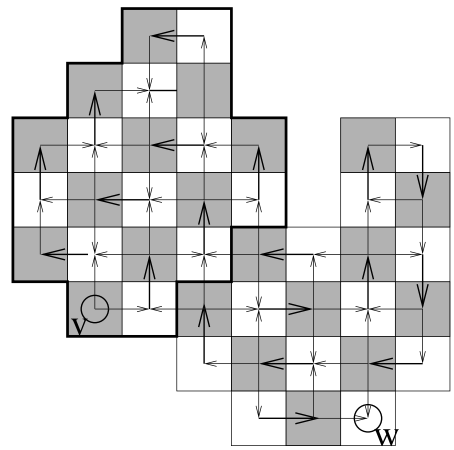


Figure 0.3: Set of reachable vertices from v.

We can also deduce the fact that no perfect matching exists from Hall's theorem by observing that the 11 black vertices in L (the enclosed region on the right of Figure 0.3) has only 10 (white) neighbors.

1-7 Consider a bipartite graph $G = (V, E)$ with bipartition (A, B) $(V = A \cup B)$. Let $\mathcal{I} = \{X \subseteq A :$ there exists a matching M of G such that all vertices of X are matched$\}$.

Show that

(a) If $X \in \mathcal{I}$ and $Y \subseteq X$ then $Y \in \mathcal{I}$.

(b) If $X, Y \in \mathcal{I}$ and $|X| < |Y|$ then there exists $y \in Y \setminus X$ such that $X \cup \{y\} \in \mathcal{I}$.

(Later in the class, we will discuss matroids, and properties (i) and (ii) form the definition of independent sets of a matroid.)

(a) Let $Y \subset X \in \mathcal{I}$. Since X is an independent set, there exists a matching M_X that covers X. This matching also covers Y. Hence Y is an independent set.

(b) Let $X, Y \in \mathcal{I}$ with $|X| < |Y|$. It follows that there exist matchings M_X and M_Y such that M_X covers X and M_Y covers Y. Consider the graph $G' = (V, M_X \Delta M_Y)$. The set of edges of G' is the union of paths and cycles.

If M_X covers some element y in $Y \setminus X$. Then $X + y$ is an independent set.

Otherwise, all the vertices in $Y \setminus X$ are of degree 1 in G'. Since $|Y| > |X|$, we have $|Y \setminus X| > |X \setminus Y|$. Therefore, by the previous observation, there are more degree 1 vertices in $Y \setminus X$ than in $X \setminus Y$. It follows that there exists a path P in the decomposition of G' starting in a vertex $y \in Y \setminus X$ and not ending in X. We conclude that $M_X \Delta P$ is a matching of G that covers $X \cup \{y\}$. Thus, $X + y$ is an independent set.

1-10 Consider a bipartite graph $G = (V, E)$ with bipartition (A, B). For $X \subseteq A$, define $\text{def}(X) = |X| - |N(X)|$ where $N(X) = \{b \in B : \exists a \in X \text{ with } (a, b) \in E\}$. Let

$$\text{def}_{max} = \max_{X \subseteq A} \text{def}(X).$$

Since $\text{def}(\emptyset) = 0$, we have $\text{def}_{max} \geq 0$.

(a) Generalize Hall's theorem by showing that the maximum size of a matching in a bipartite graph G equals $|A| - \text{def}_{max}$.

(b) For any 2 subsets $X, Y \subseteq A$, show that

$$\text{def}(X \cup Y) + \text{def}(X \cap Y) \geq \text{def}(X) + \text{def}(Y).$$

(a) Clearly, the size of a maximum matching cannot be more than $|A| - \text{def}_{max}$ (since any matching has at most $|A| - |X|$ edges incident to $A - X$ and at most $|N(X)|$ edges incident to X).

Conversely, consider the minimum vertex cover C and let $X = A \setminus C$. Observe that $N(X) \subseteq C \cap B$, and thus

$$\text{def}(X) = |X| - |N(X)| \geq |A \setminus C| - |C \cap B| = |A| - |C \cap A| - |C \cap B| = |A| - |C|.$$

Therefore $\text{def}_{max} \geq |A| - |C|$ and the result follows from König's theorem.

(b) This is a simple counting argument. First of all,

$$|X \cup Y| + |X \cap Y| = |X| + |Y|.$$

Furthermore,

$$|N(X \cup Y)| + |N(X \cap Y)| \leq |N(X)| + |N(Y)|,$$

since every vertex b in B contributes at least as much to the right-hand-side than to the left-hand-side. Indeed, if $b \in N(X \cup Y) \setminus N(X \cap Y)$, it should be either in $N(X)$ or in $N(Y)$, while if $b \in N(X \cap Y)$, it should be in both $N(X)$ and in $N(Y)$.

1-18 We have shown that there always exists a solution x to the linear program (P) with all components integral. Reprove this result in the following way.

Take a (possibly non-integral) optimum solution x^*. If there are many optimum solutions, take one with as few non-integral values x^*_{ij} as possible. Show that, if x^* is not integral, there exists a cycle C with all edges $e = (i, j) \in C$ having a non-integral value x^*_{ij}. Now show how to derive another optimum solution with fewer non-integral values, leading to a contradiction.

Let us first recall the linear program (P):

$$\begin{aligned}
&\text{Min} \quad \sum_{i,j} c_{ij} x_{ij} \\
&\text{subject to:} \\
&\quad \sum_j x_{ij} = 1 \qquad\qquad i \in A \\
&\quad \sum_i x_{ij} = 1 \qquad\qquad j \in B \\
&\quad x_{ij} \geq 0 \qquad\qquad i \in A, j \in B
\end{aligned}$$

Let x^* be an optimum solution of (P) with the fewest non-integral values x^*_{ij}. We may assume that there exists at least one ij with non-integral x^*_{ij} (otherwise, we are done). Let $G = (V, E)$ be the bipartite graph with bipartition (A, B) $(V = A \cup B)$ and the edge set

$$E = \{(i, j) \in A \times B : x^*_{ij} \text{ is not integral}\}.$$

We claim that for every $i \in V$ the degree of i with respect to G is not equal to one. Suppose not. Then there is $i \in A$ and $j' \in B$ (or change the role of A and B) such that $x^*_{ij'}$ is non-integral, and x^*_{ij} is integral for all $j \in B \setminus \{j'\}$. Since $\sum_{j \in B} x^*_{ij} = 1$, we have

$$x^*_{ij'} = 1 - \sum_{j \in B \setminus \{j'\}} x^*_{ij}.$$

This is a contradiction because the left-hand side is non-integral but the right-hand side is integral.

Now, note that every connected component of G is either an isolated vertex or a graph in which all vertices have degree at least 2. Moreover, there exists a connected component of the latter type because G has at least one edge. It implies that there exists a cycle C in G.

Let us construct an optimum solution x' with fewer non-integral values than x^*. Let $e_1, e_2, \ldots, e_\ell$ be edges of C in cyclic order. Since G is bipartite, the length ℓ of C must be an even number. Let

$$x'_{ij} = \begin{cases} x^*_{ij} - t & \text{if } ij = e_k \text{ for some odd } k \\ x^*_{ij} + t & \text{if } ij = e_k \text{ for some even } k \\ x^*_{ij} & \text{otherwise.} \end{cases}$$

Note that x' is a feasible solution as long as every entry of x' is positive, i.e.

$$- \min_{\text{odd } k} x^*_{e_k} \leq t \leq \min_{\text{even } k} x^*_{e_k}$$

Moreover, the objective function has the value

$$\sum_{ij} c_{ij}x'_{ij} = \sum_{ij} c_{ij}x^*_{ij} + t\sum_{k=1}^{\ell}(-1)^k c_{e_k}.$$

Since x^* is an optimum solution, we must have

$$\sum_{k=1}^{t}(-1)^k c_{e_k} = 0$$

because otherwise we can set t to be a value such that the objective value of x' is smaller than that of x^*. It implies that x'_{ij} is another optimum solution as long as it is feasible.

Set t to be $\min_{\text{even } k} x^*_{e_k}$. Let k' be an index such that $t = x^*_{e_{k'}}$. Then, $x'_{e_{k'}} = 0$ so x' has fewer non-integral values than x^* which contradicts the minimality of x^*.

1-11 Let $S = \{1, 2, \cdots, n\}$. Let A_k be the set of all subsets of S of cardinality k (thus $|A_k| = \binom{n}{k}$). Let $k < \frac{n}{2}$. Consider the graph G_k with bipartition A_k and A_{k+1}, and with $E = \{(a, b)|a \in A_k, b \in A_{k+1} \text{ and } a \subset b\}$.

(a) Prove that the maximum matching in G_k has size A_k (remember $k < n/2$).

(b) Prove *Sperner's lemma*. The maximum number of subsets of S such that no subset is contained into another is $\binom{n}{\lfloor n/2 \rfloor}$.

(a) Let X be a subset of A_k. Note that any vertex in A_k has degree $n - k$ in G_k. So, the number of edges between X and $N(X)$ is $(n - k)|X|$. On the other hand, the number of edges adjacent to $N(X)$ is $(k + 1)|N(X)|$ since any vertex in A_{k+1} has degree $k + 1$. Thus, $(n - k)|X| \le (k + 1)|N(X)|$. Since $k < \frac{n}{2}$, we have

$$|X| \le \frac{k+1}{n-k}|N(X)| \le |N(X)|.$$

By Hall's Theorem, there is a matching in G_k covering A_k.

(b) For a collection $\mathcal{C}$ of subsets of S, we call it a *chain* if for any $x, y \in \mathcal{C}$ either $x \subset y$ or $y \subset x$. In other words, chain is a sequence of subsets $a_1 \subset a_2 \subset \ldots \subset a_k$. On the other hand, we call a collection $\mathcal{F}$ of subsets of S an *antichain*, if no subset is contained in another. Note that any chain and antichain can share at most one element.

We claim that the collection of all subsets of S can be partitioned into $\binom{n}{\lceil n/2 \rceil}$ chains. It implies that the size of antichain is at most $\binom{n}{\lceil n/2 \rceil}$, since antichain can have at most one element from each chain.

Recall part (a). We know that G_k has a matching covering A_k if $k < \lceil \frac{n}{2} \rceil$. Similarly, if $k \geq \lceil \frac{n}{2} \rceil$ then G_k has a matching covering A_{k+1}. Let M be the union of those matchings in G_k for $k = 0, 1, \ldots, n-1$. Note that M consists of disjoint paths, and for each path there are indices k and ℓ such that the path is of the form $a_k a_{k+1} \ldots a_\ell$ where $a_j \in A_j$ for $j = k, \ldots, \ell$ and $a_j a_{j+1} \in M$. Moreover, each path contains exactly one element from $A_{\lceil \frac{n}{2} \rceil}$. Since each path is a chain, we have $\binom{n}{\lceil n/2 \rceil}$ disjoint chains covering all subsets of S.

1-15 Consider a bipartite graph $G = (V, E)$ in which every vertex has degree k (a so-called k-regular bipartite graph). Prove that such a graph always has a perfect matching in two different ways:

(a) by using König's theorem,

(b) by using the linear programming formulation we have derived in this section.

Let A, B be the bipartition of V.

(a) Because of k-regularity, we have $|A| = |B|$. Let $n = |A|$. By König's theorem, let C be a minimum vertex cover of size equal to the maximum matching. Then, $N(A \setminus C) \subseteq B \cap C$, and because of k-regularity, $|A \setminus C| \leq |B \cap C|$. Similarly, $|B \setminus C| \leq |A \cap C|$. Adding the inequalities we get $|V \setminus C| \leq |C|$, which implies that $|C| \geq |V|/2$.

(b) Any integer solution of the LP formulation

$$\begin{array}{lll} \text{Min} & \sum_{i,j} c_{ij} x_{ij} & \\ \text{subject to:} & & \\ & \sum_j x_{ij} = 1 & i \in A \\ & \sum_i x_{ij} = 1 & j \in B \\ & x_{ij} \geq 0 & i \in A, j \in B \end{array}$$

is a perfect matching. Also, all the extreme points (if any) of the LP are integral (see lecture notes on bipartite matching). Thus, it is enough to prove that the LP is feasible (so it will have at least one extreme point), and this is indeed the case as $x_{ij} = 1/k$ for all edges (i, j) is a feasible solution.

(One needs to add some assumption for the result to be true for non-bipartite graphs; indeed a cycle on 3 vertices does not have a perfect matching.)